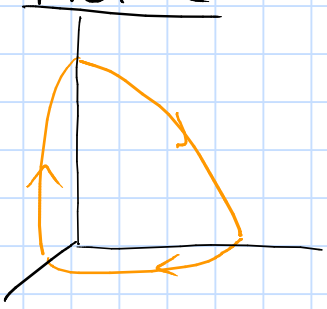
**Ex:** Compute $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F} = \langle zx+y,\ x-zy,\ x^2+y^2\rangle$ and $C$ the boundary of the part of paraboloid $z = 1-x^2-y^2$ in the first octant oriented clockwise when viewed from above.

Picture



**Sol:** We apply Stokes's Theorem, using $C = -\partial S$ for $S$ parabolic surface described above.

$$\therefore \int_C \vec{F}\cdot d\vec{r} = -\iint_S \text{Curl}(\vec{F})\cdot d\vec{S}$$

$$= -\iint_D \text{Curl}(\vec{F})(\vec{s}(r,\theta))\cdot(\vec{s}_r\times\vec{s}_\theta)\,dA$$

Use $\vec{s}(r,\theta) = \langle r\cos(\theta),\ r\sin(\theta),\ 1-r^2\rangle$ $\quad(= \langle x,y,z\rangle)$

on $D = [0,1]\times[0,\frac{\pi}{2}]$.

$\vec{s}_r = \langle \cos(\theta),\ \sin(\theta),\ -2r\rangle$ $\qquad$ $s_\theta = \langle -r\sin(\theta),\ r\cos(\theta),\ 0\rangle$

$$\vec{s}_r\times\vec{s}_\theta = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & \sin(\theta) & -2r \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{bmatrix}$$

$$= \langle \sin(\theta)\cdot 0 - (-2r)(r\cos(\theta)),\ -(\cos(\theta)\cdot 0 - (-2r)(-r\sin(\theta))),\ \cos(\theta)(r\cos(\theta)) - \sin(\theta)(-r\sin(\theta))\rangle$$

$$= \langle 2r^2\cos(\theta),\ 2r^2\sin(\theta),\ r(\cos^2(\theta)+\sin^2(\theta))\rangle$$

$$= r\langle 2r\cos(\theta), 2r\sin(\theta), 1\rangle$$

$$\text{Curl}(\vec{F}) = \nabla\times\vec{F} = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zx+y & x-zy & x^2+y^2 \end{bmatrix}$$

$$= \langle \tfrac{\partial}{\partial y}[x^2+y^2] - \tfrac{\partial}{\partial z}[x-zy],\ -(\tfrac{\partial}{\partial x}[x^2+y^2] - \tfrac{\partial}{\partial z}[zx+y]),\ \tfrac{\partial}{\partial x}[x-zy] - \tfrac{\partial}{\partial y}[zx+y]\rangle$$

$$= \langle 2y+y,\ -(2x-x),\ 1-1\rangle$$

$$= \langle 3y,\ -x,\ 0\rangle$$

$$\therefore \text{Curl}(\vec{F})(s(r,\theta)) = \langle 3r\sin(\theta),\ -r\cos(\theta),\ 0\rangle = r\langle 3\sin(\theta),\ -\cos(\theta),\ 0\rangle$$

$$\therefore \text{Curl}(\vec{F})(\vec{s}(r,\theta))\cdot(\vec{s}_r\times\vec{s}_\theta) = r^2(6r\sin(\theta)\cos(\theta) - 2r\sin(\theta)\cos(\theta) + 0)$$

$$= 4r^3\sin(\theta)\cos(\theta) = 2r^3\sin(2\theta)$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = -\int_{r=0}^{1}\int_{\theta=0}^{\pi/2} 2r^3\sin(2\theta)\,d\theta\,dr \quad \leftarrow \text{Substituting into earlier work.}$$

$$= -\int_{r=0}^{1} r^3\left[-\cos(2\theta)\right]_{\theta=0}^{\pi/2} dr$$

$$= +\int_{r=0}^{1} r^3(-1-1)\,dr$$

$$= -\frac{1}{2}\left[r^4\right]_{r=0}^{1} = -\tfrac{1}{2}(1-0) = -\frac{1}{2}$$

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